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RESEARCH ARTICLE

On the semigroup of order-preserving partial isometries of a finite chain with restricted range

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Abstract

Let I_n be the symmetric inverse semigroup on $X_n = \{1, 2, \ldots, n\}$, and let ODP_n be the subsemigroup of I_n consisting of all order-preserving partial isometries of X_n. Let $Y = \{1, 2, \ldots, r\}$ be a non-empty subset of X_n. Define

$$ODPI_{n,r} = \{\alpha \in ODP_n : \operatorname{im} \alpha \subseteq Y\}.$$

In this paper, we give a necessary and sufficient condition for $ODPI_{n,r}$ to be an inverse semigroup and characterize its Green relations. Moreover, the cardinality and the rank of $ODPI_{n,r}$ are investigated.

Keywords Partial isometries · Order-preserving · Green's relations · Rank · Injective partial transformations

Mathematics Subject Classification 20M20

1 Introduction and preliminaries

For standard terms and concepts in semigroup theory (such as regularity, Green's relations and isomorphism), we refer the reader to Howie [8].

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The *partial transformation semigroup* on a set X, denoted $P(X)$, is the collection of all functions from a subset of X into X with composition. Let $T(X)$ be the set of all transformations from X into itself. Then $T(X)$ is a subsemigroup of $P(X)$ and it is called the *full transformation semigroup* on X. It is well-known that $P(X)$ and $T(X)$ are regular semigroups. Let $I(X) = \{\alpha \in P(X) \colon \alpha \text{ is injective}\}$. Then $I(X)$ is an inverse subsemigroup of $P(X)$, called the *symmetric inverse semigroup* on X.

In this paper, we will write transformations to the right of their argument and compose them from left to right: $x(\alpha\beta) = (x\alpha)\beta$. The domain and the range of α in $P(X)$ are denoted by $\operatorname{dom}\alpha$ and $\operatorname{im}\alpha$, respectively. We denote the empty transformation by θ. For a non-empty subset A of X, we let id_A be the identity map on A. Both θ and id_A are contained in $P(X)$.

The cardinality of a set A is denoted by $|A|$. The *rank* of a semigroup S, denoted by $\operatorname{rank}(S)$, is the least size of a generating set of S. That is,

$$\operatorname{rank}(S) = \min\{|A| : A \subseteq S \text{ and } \langle A \rangle = S\}$$

where $\langle A \rangle$ is the subsemigroup of S generated by A.

Let Y be a fixed non-empty subset of X. In 1975, Symons [20] introduced and studied the subsemigroup of $T(X)$ defined by

$$T(X, Y) = \{\alpha \in T(X) \colon \operatorname{im}\alpha \subseteq Y\}.$$

We see that if $Y = X$, then $T(X, Y) = T(X)$. That is, $T(X)$ is a special case of $T(X, Y)$. In 2005, Nenthein et al. [11] characterized the regular elements of $T(X, Y)$. In 2008, Sanwong and Sommanee [15] described the largest regular subsemigroup of $T(X, Y)$. They also gave a class of maximal inverse subsemigroups of this semigroup and characterized Green's relations on $T(X, Y)$. In 2014, Fernandes and Sanwong [7] defined $PT(X, Y) = \{\alpha \in P(X) : \operatorname{im}\alpha \subseteq Y\}$ and $I(X, Y) = \{\alpha \in I(X) : \operatorname{im}\alpha \subseteq Y\}$. Then $PT(X, Y)$ and $I(X, Y)$ are subsemigroups of $P(X)$. They described the largest regular subsemigroup and determined Green's relations of $PT(X, Y)$ and $I(X, Y)$. In the case when $Y = \{1, 2, \ldots, r\}$ is a non-empty subset of a finite set $X_n = \{1, 2, \ldots, n\}$, we write $PT_{n,r}$, $T_{n,r}$ and $I_{n,r}$ for the semigroups $PT(X_n, Y)$, $T(X_n, Y)$ and $I(X_n, Y)$, respectively. The authors also computed the ranks of those semigroups.

There are other papers that study transformation semigroups with restricted range, see for example [5,9,12–14,16–19,21,22].

For a finite set $X_n = \{1, 2, \ldots, n\}$, we write I_n for $I(X_n)$. A transformation $\alpha \in I_n$ is said to be *order-preserving* if $\forall x, y \in \operatorname{dom}\alpha, x \le y \Rightarrow x\alpha \le y\alpha$. We call α an *isometry* (or *distance-preserving*) if $|x - y| = |x\alpha - y\alpha|$ for all $x, y \in \operatorname{dom}\alpha$. Let

$$DP_n = \{\alpha \in I_n \colon \forall x, y \in \operatorname{dom}\alpha, |x - y| = |x\alpha - y\alpha|\}$$

be the subsemigroup of I_n consisting of all partial isometries of X_n. Also, let

$$ODP_n = \{\alpha \in DP_n \colon \forall x, y \in \operatorname{dom}\alpha, x \le y \Rightarrow x\alpha \le y\alpha\}$$

be the subsemigroup of DP_n consisting of all order-preserving partial isometries of X_n. Then DP_n and ODP_n are inverse subsemigroups of I_n. Notice that the semigroups

I_n, DP_n and ODP_n contain θ and id_A for any non-empty subset A of X_n. In 2013, Dimitrova [4] proved that the semigroup ODP_n has exactly $n+1$ maximal subsemigroups. In 2014, Al-Kharousi et al. [1] showed that $|DP_n| = 3 \cdot 2^{n+1} - (n+2)^2 - 1$ and $|ODP_n| = 3 \cdot 2^n - 2(n+1)$. Later in 2016, Al-Kharousi et al. [2] investigated the structure of a partial isometry and characterized Green's relations on DP_n and ODP_n. They also proved that for $n \geq 2$, $\mathrm{rank}(ODP_n) = n+1$ and $\mathrm{rank}(DP_n) = \lfloor \frac{n+3}{2} \rfloor$ where $\lfloor x \rfloor$ is the floor function. In the same year, Fernandes and Quinteiro [6] gave presentations for the semigroups DP_n and ODP_n. In 2018, Namnak et al. [10] studied the natural partial orders on DP_n and ODP_n. Latest in 2018, Bugay et al. [3] defined the subsemigroups

$$DP_{n,r} = \{\alpha \in DP_n : |\mathrm{im}\alpha| \leq r\} \text{ and } ODP_{n,r} = \{\alpha \in ODP_n : |\mathrm{im}\alpha| \leq r\}$$

of DP_n where $2 \leq r \leq n-1$. The authors of [3] also calculated the cardinalities of $DP_{n,r}$ and $ODP_{n,r}$. Moreover, they found the ranks of the subsemigroups $DP_{n,r}$ and $ODP_{n,r}$. Indeed, $\mathrm{rank}(DP_{n,r}) = \mathrm{rank}(ODP_{n,r}) = \binom{n}{r}$.

Let $A = \{a_1, a_2, \ldots, a_k\} \subseteq X_n$ such that $a_1 < a_2 < \cdots < a_k$ and $k \geq 2$. We define the *gap* of the set A as the following *ordered* $(k-1)$-*tuple*:

$$g(A) = (a_2 - a_1, a_3 - a_2, \ldots, a_k - a_{k-1}).$$

In the case $A = \{a\} \subseteq X_n$, we let $g(A) = (0)$. Notice that for $\alpha \in ODP_n$, $g(\mathrm{dom}\,\alpha) = g(\mathrm{im}\,\alpha)$.

Lemma 1.1 ([1, Lemma 3.3]) *Let $A \subseteq X_n$ with $|A| = p \geq 1$. If $g(A) = (d_1, d_2, \ldots, d_{p-1})$, then $p - 1 \leq \sum_{i=1}^{p-1} d_i \leq n-1$.*

From [2], Green's relations $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{D}$ on ODP_n can be characterized as follows:

Lemma 1.2 ([2, Lemma 2.1, Remark 2.2, Theorem 2.5]) *Let $\alpha, \beta \in ODP_n$. Then*

(1) $\alpha \mathcal{L} \beta$ if and only if $\mathrm{im}\,\alpha = \mathrm{im}\,\beta$,
(2) $\alpha \mathcal{R} \beta$ if and only if $\mathrm{dom}\,\alpha = \mathrm{dom}\,\beta$,
(3) $\alpha \mathcal{H} \beta$ if and only if $\alpha = \beta$,
(4) $\alpha \mathcal{D} \beta$ if and only if $g(\mathrm{im}\,\alpha) = g(\mathrm{im}\,\beta)$.

For convenience,

$$\alpha = \begin{pmatrix} x_1 & x_2 & \cdots & x_k \\ a_1 & a_2 & \cdots & a_k \end{pmatrix} \tag{1.1}$$

means $\alpha \in I_n$ such that $x_i\alpha = a_i$ for all $1 \leq i \leq k$, $\mathrm{dom}\,\alpha = \{x_1, x_2, \ldots, x_k\}$ and $\mathrm{im}\,\alpha = \{a_1, a_2, \ldots, a_k\}$ where $x_1 < x_2 < \cdots < x_k$. Furthermore, if such α is order-preserving, then we have $a_1 < a_2 < \cdots < a_k$.

For $n \geq 2$, let

$$\eta = \begin{pmatrix} 2 & 3 & \cdots & n \\ 1 & 2 & \cdots & n-1 \end{pmatrix} \text{ and } G = \{\eta, \eta^{-1}\} \cup \{\mathrm{id}_{X_n \setminus \{i\}} : 2 \leq i \leq n-1\}.$$

Then we have the following lemma.

Lemma 1.3 *([2, Theorem 3.1]) For $n \geq 2$, we have* $\operatorname{rank}(ODP_n) = n + 1$ *where* $G \cup \{\mathrm{id}_{X_n}\}$ *is the unique minimum generating set for* ODP_n.

Throughout this paper, we fix $Y = \{1, 2, \ldots, r\} \subseteq X_n = \{1, 2, \ldots, n\}$ and let

$$ODPI_{n,r} = ODP_n \cap I_{n,r}.$$

It is clear that

$$ODPI_{n,r} = \{\alpha \in ODP_n : \operatorname{im}\alpha \subseteq Y\}.$$

Since $\theta \in ODP_n \cap I_{n,r}$, we obtain $ODPI_{n,r} = ODP_n \cap I_{n,r} \neq \emptyset$. Thus, $ODPI_{n,r}$ is a subsemigroup of ODP_n and $I_{n,r}$. Moreover, if $n = r$ (i.e., $X_n = Y$), then $ODPI_{n,r} = ODP_n \cap I_{n,n} = ODP_n \cap I_n = ODP_n$. We may regard $ODPI_{n,r}$ as a generalization of ODP_n. Note that $ODPI_{n,r} \subseteq ODP_{n,r}$ as defined in [3] since $|\operatorname{im}\alpha| \leq |Y| = r$ for all $\alpha \in ODPI_{n,r}$.

In this paper, we give a necessary and sufficient condition for $ODPI_{n,r}$ to be an inverse subsemigroup of ODP_n and characterize Green's relations on $ODPI_{n,r}$. Also, we investigate the cardinality of $ODPI_{n,r}$ and compute its rank.

2 The semigroup $ODPI_{n,r}$

We begin this section with the following remark which will be used throughout the paper.

Remark 2.1 Let $\alpha \in I_{n,r} \setminus \{\theta\}$ be order-preserving and $\operatorname{dom}\alpha = \{x_1, x_2, \ldots, x_k\}$. Then $\alpha \in ODPI_{n,r}$ if and only if $|x_{i+1} - x_i| = |x_{i+1}\alpha - x_i\alpha|$ for all $1 \leq i \leq k-1$.

With the notation (1.1) and Remark 2.1, for any $\theta \neq \alpha \in ODPI_{n,r}$, we can write

$$\alpha = \begin{pmatrix} x_1 & x_2 & \cdots & x_k \\ a_1 & a_2 & \cdots & a_k \end{pmatrix}$$

where $\operatorname{dom}\alpha = \{x_1, x_2, \ldots, x_k\} \subseteq X_n$ and $\operatorname{im}\alpha = \{a_1, a_2, \ldots, a_k\} \subseteq Y$ such that $x_1 < x_2 < \cdots < x_k$, $a_1 < a_2 < \cdots < a_k$ and $|x_{i+1} - x_i| = |a_{i+1} - a_i|$ for all $1 \leq i \leq k-1$.

Theorem 2.1 *$ODPI_{n,r}$ is an inverse semigroup if and only if $X_n = Y$.*

Proof If $X_n = Y$, then $ODPI_{n,r} = ODP_n$ is an inverse semigroup. Conversely, assume that $ODPI_{n,r}$ is an inverse semigroup and $Y \subsetneq X_n$. Define $\alpha = \binom{n}{1}$. It is clear that $\alpha \in ODPI_{n,r}$. Since $ODPI_{n,r}$ is an inverse semigroup, there is $\beta \in ODPI_{n,r}$ such that $\alpha = \alpha\beta\alpha$. Thus, $n\alpha = n\alpha\beta\alpha = (1\beta)\alpha$, which implies that $1\beta = n \notin Y$ (since α is injective) and this leads to a contradiction. Therefore, $X_n = Y$. □

The following corollary is directly obtained from Theorem 2.1.

Corollary 2.2 *If $r < n$, then $ODPI_{n,r}$ is not isomorphic to ODP_m for any set X_m.*

Theorem 2.3 *ODP_r is the largest regular subsemigroup of $ODPI_{n,r}$. In particular, ODP_r is the largest inverse subsemigroup of $ODPI_{n,r}$.*

Proof It is known that ODP_r is regular. Since $ODP_r \subseteq ODPI_{n,r}$, we obtain ODP_r is a regular subsemigroup of $ODPI_{n,r}$. On the other hand, let α be a regular element of $ODPI_{n,r}$ and take $\beta \in ODPI_{n,r}$ such that $\alpha = \alpha\beta\alpha$. We write

$$\alpha = \begin{pmatrix} x_1 & x_2 & \cdots & x_k \\ a_1 & a_2 & \cdots & a_k \end{pmatrix}.$$

Then $x_i\alpha = (x_i\alpha)\beta\alpha = (a_i\beta)\alpha$ and so $x_i = a_i\beta \in Y$ (since α is injective). Hence, $\operatorname{dom}\alpha \subseteq Y$, that is $\alpha \in ODP_r$. It follows that ODP_r is the set of all regular elements of $ODPI_{n,r}$. Since ODP_r is an inverse subsemigroup of $ODPI_{n,r}$, this implies that ODP_r is the largest inverse subsemigroup of $ODPI_{n,r}$. □

As the idempotents of $I(X)$ are all the maps of the form id_A with $A \subseteq X$. From that fact that the idempotents of I_r and $ODPI_{n,r}$ coincide, we obtain the following proposition.

Proposition 2.4 *The idempotents of $ODPI_{n,r}$ are precisely the elements of the set $\{\mathrm{id}_A : \emptyset \neq A \subseteq Y\} \cup \{\theta\}$.*

Proposition 2.5 *Let $Y_1 = \{1, 2, \ldots, r_1\}$ and $Y_2 = \{1, 2, \ldots, r_2\}$ be non-empty subsets of X_n. Then $ODPI_{n,r_1}$ is isomorphic to $ODPI_{n,r_2}$ if and only if $r_1 = r_2$.*

Proof Suppose that $ODPI_{n,r_1}$ is isomorphic to $ODPI_{n,r_2}$ and $r_1 \neq r_2$. We may assume that $r_1 < r_2$, that is, $Y_1 \subsetneq Y_2$. It follows that $ODPI_{n,r_1} \subseteq ODPI_{n,r_2}$. Since $|ODPI_{n,r_1}| = |ODPI_{n,r_2}|$ is finite, we obtain $ODPI_{n,r_1} = ODPI_{n,r_2}$. This implies that $Y_1 = Y_2$, a contradiction. Therefore, $r_1 = r_2$. The converse is clear. □

3 Green's relations

Our aim in this section is to describe Green's relations on the semigroups $ODPI_{n,r}$.

Lemma 3.1 *Let $\alpha, \beta \in ODPI_{n,r}$. If $\alpha\mathcal{L}\beta$ on $ODPI_{n,r}$, then $\alpha, \beta \in ODP_r$ or $\alpha = \beta$.*

Proof Assume that $\alpha\mathcal{L}\beta$ on $ODPI_{n,r}$. Then there exist $\gamma, \lambda \in ODPI_{n,r}^1$ such that $\alpha = \gamma\beta$ and $\beta = \lambda\alpha$. If $\gamma = 1$ or $\lambda = 1$, then $\alpha = \beta$. Now, suppose that $\gamma \neq 1 \neq \lambda$, that is $\gamma, \lambda \in ODPI_{n,r}$. Let $x \in \operatorname{dom}\alpha$. Then $x\alpha = x\gamma\beta = x\gamma(\lambda\alpha) = (x\gamma\lambda)\alpha$, this implies that $x = x\gamma\lambda \in Y$. Thus, $\operatorname{dom}\alpha \subseteq Y$ and so $\alpha \in ODP_r$. Similarly, we can show that $\beta \in ODP_r$. □

Theorem 3.2 *Let $\alpha, \beta \in ODPI_{n,r}$. Then $\alpha\mathcal{L}\beta$ on $ODPI_{n,r}$ if and only if ($\alpha, \beta \in ODP_r$ and $\operatorname{im}\alpha = \operatorname{im}\beta$) or ($\alpha, \beta \notin ODP_r$ and $\alpha = \beta$).*

Proof Assume that $\alpha\mathcal{L}\beta$ on $ODPI_{n,r}$. By Lemma 3.1, we have $\alpha, \beta \in ODP_r$ or $\alpha = \beta$. By [8, Proposition 2.4.2], if $\alpha, \beta \in ODP_r$, then $\alpha\mathcal{L}\beta$ on ODP_r since ODP_r is a regular subsemigroup of $ODPI_{n,r}$. Hence, $\operatorname{im}\alpha = \operatorname{im}\beta$ by Lemma 1.2. If $\alpha \notin ODP_r$ or $\beta \notin ODP_r$, then $\alpha = \beta \notin ODP_r$.

Conversely, assume the conditions hold. If $\alpha = \beta$, then we obtain $\alpha\mathcal{L}\beta$ on $ODPI_{n,r}$. If $\alpha, \beta \in ODP_r$ and $\operatorname{im}\alpha = \operatorname{im}\beta$, then $\alpha\mathcal{L}\beta$ on ODP_r by Lemma1.2. So, we automatically have $\alpha\mathcal{L}\beta$ on $ODPI_{n,r}$. □

Theorem 3.3 *Let $\alpha, \beta \in ODPI_{n,r}$. Then $\alpha\mathcal{R}\beta$ on $ODPI_{n,r}$ if and only if $\operatorname{dom}\alpha = \operatorname{dom}\beta$.*

Proof If $\alpha\mathcal{R}\beta$ on $ODPI_{n,r}$, then $\alpha\mathcal{R}\beta$ on ODP_n and so $\operatorname{dom}\alpha = \operatorname{dom}\beta$ by Lemma 1.2. On the other hand, assume that $\operatorname{dom}\alpha = \operatorname{dom}\beta$. If $\alpha = \theta$ or $\beta = \theta$, then $\operatorname{dom}\alpha = \emptyset = \operatorname{dom}\beta$ and so $\alpha = \theta = \beta$. Thus, $\alpha\mathcal{R}\beta$ on $ODPI_{n,r}$. Now, suppose that α and β are not the empty map. We can write

$$\alpha = \begin{pmatrix} x_1 & x_2 & \cdots & x_k \\ a_1 & a_2 & \cdots & a_k \end{pmatrix} \text{ and } \beta = \begin{pmatrix} x_1 & x_2 & \cdots & x_k \\ b_1 & b_2 & \cdots & b_k \end{pmatrix}.$$

Define

$$\gamma = \begin{pmatrix} b_1 & b_2 & \cdots & b_k \\ a_1 & a_2 & \cdots & a_k \end{pmatrix} \in ODPI_{n,r} \text{ and } \lambda = \begin{pmatrix} a_1 & a_2 & \cdots & a_k \\ b_1 & b_2 & \cdots & b_k \end{pmatrix} \in ODPI_{n,r}.$$

It is easy to verify that $\alpha = \beta\gamma$ and $\beta = \alpha\lambda$. Therefore, $\alpha\mathcal{R}\beta$ on $ODPI_{n,r}$. □

Remark 3.1 Let $\alpha, \beta \in ODPI_{n,r}$ and $\alpha\mathcal{R}\beta$ on $ODPI_{n,r}$. If $\alpha \in ODP_r$, then $\beta \in ODP_r$. In fact, if $\alpha \in ODP_r$, then $\operatorname{dom}\beta = \operatorname{dom}\alpha \subseteq Y$ by Theorem 3.3. Thus, $\beta \in ODP_r$. It follows that if $\alpha\mathcal{R}\beta$ on $ODPI_{n,r}$, then $\alpha, \beta \in ODP_r$ or $\alpha, \beta \notin ODP_r$.

Lemma 3.4 *Let $\alpha, \beta \in ODPI_{n,r}$ and $\alpha\mathcal{D}\beta$ on $ODPI_{n,r}$. Then either $\alpha, \beta \in ODP_r$ or $\alpha, \beta \notin ODP_r$.*

Proof Assume that $\alpha \in ODP_r$. Since $\alpha\mathcal{D}\beta$ on $ODPI_{n,r}$, there exists $\gamma \in ODPI_{n,r}$ such that $\alpha\mathcal{L}\gamma\mathcal{R}\beta$. From $\alpha\mathcal{L}\gamma$ and $\alpha \in ODP_r$, we get $\gamma \in ODP_r$ by Lemma 3.1. Then by Remark 3.1, we obtain $\beta \in ODP_r$ since $\gamma\mathcal{R}\beta$ and $\gamma \in ODP_r$. We now apply this argument again, with α replaced by β, to obtain if $\beta \in ODP_r$, then $\alpha \in ODP_r$. Equivalently, if $\alpha \notin ODP_r$, then $\beta \notin ODP_r$. □

Theorem 3.5 *Let $\alpha, \beta \in ODPI_{n,r}$. Then $\alpha\mathcal{D}\beta$ on $ODPI_{n,r}$ if and only if $\big(\alpha, \beta \in ODP_r$ and $g(\operatorname{im}\alpha) = g(\operatorname{im}\beta)\big)$ or $\big(\alpha, \beta \notin ODP_r$ and $\operatorname{dom}\alpha = \operatorname{dom}\beta\big)$.*

Proof Assume that $\alpha\mathcal{D}\beta$ on $ODPI_{n,r}$. We note that either $\alpha, \beta \in ODP_r$ or $\alpha, \beta \notin ODP_r$ by Lemma 3.4. Since $\alpha\mathcal{D}\beta$ on $ODPI_{n,r}$, there exists $\gamma \in ODPI_{n,r}$ such that $\alpha\mathcal{L}\gamma\mathcal{R}\beta$. If $\alpha \in ODP_r$, then $\gamma \in ODP_r$ and $\operatorname{im}\alpha = \operatorname{im}\gamma$ by Theorem 3.2. Since $\gamma\mathcal{R}\beta$, we obtain $\operatorname{dom}\gamma = \operatorname{dom}\beta$ by Theorem 3.3. Thus, $g(\operatorname{im}\alpha) = g(\operatorname{im}\gamma) = g(\operatorname{dom}\gamma) = g(\operatorname{dom}\beta) = g(\operatorname{im}\beta)$. Now, suppose that $\alpha \notin ODP_r$. By Theorem 3.2, we have $\gamma \notin ODP_r$ since $\alpha\mathcal{L}\gamma$. From $\alpha, \gamma \notin ODP_r$ and $\alpha\mathcal{L}\gamma\mathcal{R}\beta$ on $ODPI_{n,r}$, we obtain

$\alpha = \gamma$ and $\operatorname{dom}\gamma = \operatorname{dom}\beta$ by Theorem 3.2 and Theorem 3.3, respectively. Thus, $\operatorname{dom}\alpha = \operatorname{dom}\gamma = \operatorname{dom}\beta$.

Conversely, assume the conditions hold. If $\alpha, \beta \in ODP_r$ and $g(\operatorname{im}\alpha) = g(\operatorname{im}\beta)$, then $\alpha\mathcal{D}\beta$ on ODP_r by Lemma 1.2. It follows immediately that $\alpha\mathcal{D}\beta$ on $ODPI_{n,r}$. In the later case, if $\operatorname{dom}\alpha = \operatorname{dom}\beta$, then $\alpha\mathcal{R}\beta$ on $ODPI_{n,r}$ by Theorem 3.3. Therefore, $\alpha\mathcal{D}\beta$ on $ODPI_{n,r}$ since $\mathcal{R} \subseteq \mathcal{D}$. □

Remark 3.2 $\mathcal{J} = \mathcal{D}$ on $ODPI_{n,r}$ because the semigroup $ODPI_{n,r}$ is finite.

Theorem 3.6 *Let $\alpha, \beta \in ODPI_{n,r}$. Then $\alpha\mathcal{H}\beta$ on $ODPI_{n,r}$ if and only if $\alpha = \beta$.*

Proof Suppose that $\alpha\mathcal{H}\beta$ on $ODPI_{n,r}$, that is, $\alpha\mathcal{L}\beta$ and $\alpha\mathcal{R}\beta$. Then ($\alpha, \beta \in ODP_r$ and $\operatorname{im}\alpha = \operatorname{im}\beta$) or ($\alpha, \beta \notin ODP_r$ and $\alpha = \beta$), and $\operatorname{dom}\alpha = \operatorname{dom}\beta$ by Theorem 3.2 and 3.3, respectively. We get ($\alpha, \beta \in ODP_r$, $\operatorname{im}\alpha = \operatorname{im}\beta$ and $\operatorname{dom}\alpha = \operatorname{dom}\beta$) or ($\alpha, \beta \notin ODP_r$ and $\alpha = \beta$). It follows that $\alpha\mathcal{H}\beta$ on ODP_r or $\alpha = \beta$. Thus, $\alpha = \beta$ by Lemma 1.2. The converse is clear. □

4 The cardinality and ranks

For any positive integer number n, the authors of [1] defined

$$F(n; p) = |\{\alpha \in ODP_n : |\operatorname{im}\alpha| = p\}|$$

and showed that

$$F(n; p) = \frac{2n - p + 1}{p + 1}\binom{n}{p} \tag{4.1}$$

where $1 \leq p \leq n$, see [1, Proposition 2.4].

Lemma 4.1 ([1, Lemma 2.5]) For $n \geq 1$, $\sum_{k=1}^{n} \frac{2n-k+1}{k+1}\binom{n}{k} = 3 \cdot 2^n - 2n - 3$.

Lemma 4.2 ([1, Theorem 2.6]) For $n \geq 1$, $|ODP_n| = 3 \cdot 2^n - 2(n + 1)$.

From now on, let $1 \leq k \leq r \leq n$. Define

$$Z_k = \{\alpha \in ODPI_{n,r} : |\operatorname{im}\alpha| = k\} \text{ and } F(n; r; k) = |Z_k|.$$

It is easy to see that $\theta \notin Z_k$ and $F(r; k) \leq F(n; r; k)$.

For each $\alpha, \beta \in Z_k$, we define the relation $\sim$ on Z_k by

$$\alpha \sim \beta \text{ if and only if } g(\operatorname{im}\alpha) = g(\operatorname{im}\beta).$$

Then $\sim$ is an equivalence relation on Z_k. Let $\mathcal{C}_k = \{P_1, P_2, \ldots, P_t\}$ be the partition of Z_k induced by the relation $\sim$. That is, if $\alpha, \beta \in Z_k$, then

$$\alpha, \beta \in P_i \text{ if and only if } g(\operatorname{im}\alpha) = g(\operatorname{im}\beta)$$

for all $1 \leq i \leq t$.

We now describe a structure of P_i $(1 \leq i \leq t)$. Assume that $1 \leq i \leq t$ and $(d_1, d_2, \ldots, d_{k-1})$ is a gap of all elements of P_i. Let $\omega_i = \sum_{j=1}^{k-1} d_j$. Observe that $1 + \omega_i = 1 + \sum_{j=1}^{k-1} d_j \leq r$ by using Lemma 1.1. We construct

$$\xi_i = \begin{pmatrix} 1 & 1+d_1 & 1+(d_1+d_2) & \cdots & 1+(\omega_i - d_{k-1}) & 1+\omega_i \\ 1 & 1+d_1 & 1+(d_1+d_2) & \cdots & 1+(\omega_i - d_{k-1}) & 1+\omega_i \end{pmatrix} \in P_i \cap ODP_r.$$

Let V_1 and V_2 be the sets of all elements α in P_i with $\operatorname{dom}\alpha = \operatorname{dom}\xi_i$ and $\operatorname{im}\alpha = \operatorname{im}\xi_i$, respectively. Then

$$V_1 = \{\xi_i\} \cup \{\tau_q : 1 \leq q \leq r - (1+\omega_i)\} \text{ and } V_2 = \{\xi_i\} \cup \{\delta_j : 1 \leq j \leq n - (1+\omega_i)\},$$

where

$$\tau_q = \begin{pmatrix} 1 & 1+d_1 & \cdots & 1+(\omega_i - d_{k-1}) & 1+\omega_i \\ 1+q & 1+d_1+q & \cdots & 1+(\omega_i - d_{k-1})+q & 1+\omega_i+q \end{pmatrix}$$

for $1 \leq q \leq r - (1+\omega_i)$, and

$$\delta_j = \begin{pmatrix} 1+j & 1+d_1+j & 1+(d_1+d_2)+j & \cdots & 1+(\omega_i - d_{k-1})+j & 1+\omega_i+j \\ 1 & 1+d_1 & 1+(d_1+d_2) & \cdots & 1+(\omega_i - d_{k-1}) & 1+\omega_i \end{pmatrix}$$

for $1 \leq j \leq n-(1+\omega_i)$. Notice that $V_1 \subseteq ODP_r$, $\{\delta_j : 1 \leq j \leq r-(1+\omega_i)\} \subseteq ODP_r$ and $\{\delta_{r-(1+\omega_i)+p} : 1 \leq p \leq n-r\} \subseteq P_i \setminus ODP_r$. By the definition of δ_j, we can write $\delta_{r-(1+\omega_i)}$ and $\delta_{r-(1+\omega_i)+p}$ as follows:

$$\delta_{r-(1+\omega_i)} = \begin{pmatrix} r-\omega_i & r-\omega_i+d_1 & \cdots & r-d_{k-1} & r \\ 1 & 1+d_1 & \cdots & 1+(\omega_i - d_{k-1}) & 1+\omega_i \end{pmatrix}$$

and

$$\delta_{r-(1+\omega_i)+p} = \begin{pmatrix} (r-\omega_i)+p & (r-\omega_i+d_1)+p & \cdots & (r-d_{k-1})+p & r+p \\ 1 & 1+d_1 & \cdots & 1+(\omega_i - d_{k-1}) & 1+\omega_i \end{pmatrix}$$

for all $1 \leq p \leq n-r$. For an element $\gamma \in P_i \setminus ODP_r$, we have

$$\gamma = \begin{pmatrix} (r-\omega_i)+p & (r-\omega_i+d_1)+p & \cdots & (r-d_{k-1})+p & r+p \\ 1+q & 1+d_1+q & \cdots & 1+(\omega_i - d_{k-1})+q & 1+\omega_i+q \end{pmatrix}$$

for some $1 \leq p \leq n-r$ and $0 \leq q \leq r-(1+\omega_i)$. We see that $\operatorname{dom}\gamma = \operatorname{dom}\delta_{r-(1+\omega_i)+p}$ and $\operatorname{im}\gamma = \operatorname{im}\tau_q$ where $1 \leq p \leq n-r$ and $0 \leq q \leq r-(1+\omega_i)$.

From the above statements, we obtain the following configuration for P_i of Z_k.

$\delta_{n-(1+\omega_i)}$		$\cdots$		$\cdots$	
$\vdots$	$\vdots$		$\vdots$		$\vdots$
$\delta_{r-(1+\omega_i)+p}$		$\cdots$	γ	$\cdots$	
$\vdots$	$\vdots$		$\vdots$		$\vdots$
$\delta_{r-(1+\omega_i)+1}$					
$\delta_{r-(1+\omega_i)}$		$\cdots$		$\cdots$	
$\vdots$	$\vdots$		$\vdots$		$\vdots$
δ_1		$\cdots$		$\cdots$	
ξ_i	τ_1	$\cdots$	τ_q	$\cdots$	$\tau_{r-(1+\omega_i)}$

By the above configuration, all elements of $P_i \cap ODP_r$ are shaded gray and all elements of $P_i \setminus ODP_r$ are shaded white. Moreover, $|P_i \setminus ODP_r| = (n-r)(r-\omega_i)$ and the number of distinct images of all elements of P_i is equal to $|V_1| = r - \omega_i$. This implies that the number of distinct images of all elements of $Z_k = \bigcup_{i=1}^{t} P_i$ is equal to $\sum_{i=1}^{t}(r-\omega_i)$. But it is clear that the number of image sets in Y of cardinality k is $\binom{r}{k}$. Thus, we conclude that

$$\sum_{i=1}^{t}(r-\omega_i) = \binom{r}{k}. \tag{4.2}$$

Notice that $\bigcup_{i=1}^{t}(P_i \cap ODP_r) = Z_k \cap ODP_r = \{\alpha \in ODP_r : |\text{im}\,\alpha| = k\}$. Hence,

$$\sum_{i=1}^{t}|P_i \cap ODP_r| = \Big|\bigcup_{i=1}^{t}(P_i \cap ODP_r)\Big| = |\{\alpha \in ODP_r : |\text{im}\,\alpha| = k\}| = F(r;k).$$

Proposition 4.3 *For* $1 \le k \le r \le n$,

$$F(n;r;k) = \Big(\frac{2r-k+1}{k+1} + n - r\Big)\binom{r}{k}.$$

Proof We first note that $Z_k = \bigcup_{i=1}^{t} P_i = \bigcup_{i=1}^{t}\Big[(P_i \cap ODP_r) \cup (P_i \setminus ODP_r)\Big]$. Then by using (4.1) and (4.2) , we get

$$\begin{aligned}
F(n;r;k) &= |Z_k| \\
&= \sum_{i=1}^{t} |P_i \cap ODP_r| + \sum_{i=1}^{t} |P_i \setminus ODP_r| \\
&= F(r;k) + \sum_{i=1}^{t}(n-r)(r-\omega_i) \\
&= F(r;k) + (n-r)\sum_{i=1}^{t}(r-\omega_i) \\
&= \frac{2r-k+1}{k+1}\binom{r}{k} + (n-r)\binom{r}{k} \\
&= \Big(\frac{2r-k+1}{k+1} + n - r\Big)\binom{r}{k}.
\end{aligned}$$

□

Recall that $\sum_{k=1}^{r}\binom{r}{k} = 2^r - 1$ and $\sum_{k=1}^{r}\frac{2r-k+1}{k+1}\binom{r}{k} = 3\cdot 2^r - 2r - 3$ by Lemma 4.1.

Theorem 4.4 *For* $1 \le r \le n$,

$$|ODPI_{n,r}| = (n-r+3)2^r - (n+r) - 2.$$

Proof Consider

$$\begin{aligned}
\sum_{k=1}^{r} F(n;r;k) &= \sum_{k=1}^{r}\Big(\frac{2r-k+1}{k+1} + n - r\Big)\binom{r}{k} \quad \text{by Proposition 4.3} \\
&= \sum_{k=1}^{r}\frac{2r-k+1}{k+1}\binom{r}{k} + (n-r)\sum_{k=1}^{r}\binom{r}{k} \\
&= (3\cdot 2^r - 2r - 3) + (n-r)(2^r - 1) \\
&= (n-r+3)2^r - n - r - 3.
\end{aligned}$$

Then by adding the empty map θ, we get $|ODPI_{n,r}| = 1 + \sum_{k=1}^{r} F(n;r;k) = (n-r+3)2^r - (n+r) - 2$. □

Corollary 4.5 *For* $n \ge 1$, $|ODP_n| = 3\cdot 2^n - 2(n+1)$.

Proof By taking $X_n = Y$, we obtain $ODPI_{n,r} = ODP_n$ and $|ODP_n| = (n-n+3)2^n - (n+n) - 2 = 3\cdot 2^n - 2(n+1)$. □

Now, we find the rank of $ODPI_{n,r}$.

Notice that if $n = r$, then $ODPI_{n,r} = ODP_n$ and $\operatorname{rank}(ODP_n) = n + 1$. If $n = 1$, then $ODPI_{1,r} = ODPI_{1,1} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \theta \right\}$ and $\operatorname{rank}(ODPI_{1,r}) = 2$. And if $r = 1$, then

$$ODPI_{n,1} = \{\theta\} \cup \left\{ \begin{pmatrix} i \\ 1 \end{pmatrix} : 1 \leq i \leq n \right\} = \left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \ldots, \begin{pmatrix} n \\ 1 \end{pmatrix} \right\rangle.$$

Thus, $\operatorname{rank}(ODPI_{n,1}) = n$.

Next, we consider $n \geq 3$ and $2 \leq r \leq n - 1$.

For the case $k = r$, we have $\mathcal{C}_r = \{Z_r\}$ is a partition of Z_r and the image of all elements of Z_r is Y. Indeed, the gap of all elements of Z_r is the $(r - 1)$-tuple $(1, 1, \ldots, 1)$. We see that $Z_r = \{\mathrm{id}_Y\} \cup \{\delta_j : 1 \leq j \leq n - r\}$ where

$$\delta_j = \begin{pmatrix} 1+j & 2+j & \cdots & r-1+j & r+j \\ 1 & 2 & \cdots & r-1 & r \end{pmatrix}$$

for $1 \leq j \leq n - r$, and $|Z_r| = n - r + 1$. Notice that for any element $\alpha \in Z_r$,

$$1 \in \operatorname{dom} \alpha \text{ if and only if } \alpha = \mathrm{id}_Y.$$

Furthermore, $ODPI_{n,r}$ has precisely $n - r + 1$ distinct $\mathcal{R}$-classes in Z_r. Let R_j be an $\mathcal{R}$-class in Z_r $(0 \leq j \leq n - r)$. We may assume that $R_0 = \{\mathrm{id}_Y\}$ and $R_j = \{\delta_j\}$ where $1 \leq j \leq n - r$.

Lemma 4.6 *Let A be any generating set of $ODPI_{n,r}$. Then $Z_r \subseteq A$.*

Proof Let $\alpha \in Z_r$. Then $\alpha \in R_j$ for some $0 \leq j \leq n - r$ and we can write $\alpha = \alpha_1\alpha_2 \cdots \alpha_p$ for some $\alpha_1, \alpha_2, \ldots, \alpha_p \in A$. It follows that $\operatorname{dom} \alpha \subseteq \operatorname{dom} \alpha_1$ and so $r = |\operatorname{dom} \alpha| \leq |\operatorname{dom} \alpha_1| \leq r$. Whence, $|\operatorname{dom} \alpha| = r = |\operatorname{dom} \alpha_1|$. Since $\operatorname{dom} \alpha \subseteq \operatorname{dom} \alpha_1$ and $|\operatorname{dom} \alpha| = |\operatorname{dom} \alpha_1|$ is finite, we obtain $\operatorname{dom} \alpha = \operatorname{dom} \alpha_1$. Hence, $\alpha_1 \mathcal{R} \alpha$ on $OPDI_{n,r}$ by Theorem 3.3. Thus, $\alpha, \alpha_1 \in R_j$. But $|R_j| = 1$, we get $\alpha = \alpha_1 \in A$. □

Lemma 4.7 *Let $\alpha \in ODPI_{n,r} \setminus \{\theta\}$ and $\alpha = \alpha_1\alpha_2 \cdots \alpha_j \cdots \alpha_p$ where $1 \leq j \leq p$. If $1 \in \operatorname{dom} \alpha$ and $\alpha_1, \alpha_2, \ldots, \alpha_j \in Z_r$, then $\alpha_1 = \alpha_2 = \cdots = \alpha_j = \mathrm{id}_Y$.*

Proof Assume that $1 \in \operatorname{dom} \alpha$ and $\alpha_1, \alpha_2, \ldots, \alpha_j \in Z_r$. Then $1 \in \operatorname{dom} \alpha = \operatorname{dom}(\alpha_1\alpha_2 \cdots \alpha_p) \subseteq \operatorname{dom}(\alpha_1)$. Since $1 \in \operatorname{dom} \alpha_1$ and $\alpha_1 \in Z_r$, we obtain $\alpha_1 = \mathrm{id}_Y$. Now, we have $\alpha = \mathrm{id}_Y\alpha_2\alpha_3 \cdots \alpha_p$. It follows that $1\alpha = (1\mathrm{id}_Y)\alpha_2\alpha_3 \cdots \alpha_p = 1(\alpha_2\alpha_3 \cdots \alpha_p)$. Since $1 \in \operatorname{dom}(\alpha_2\alpha_3 \cdots \alpha_p) \subseteq \operatorname{dom} \alpha_2$ and $\alpha_2 \in Z_r$, this implies that $\alpha_2 = \mathrm{id}_Y$. Continuing in this way, we get $\alpha_1 = \alpha_2 = \cdots = \alpha_j = \mathrm{id}_Y$ as required. □

The partition of Z_{r-1} induced by the relation $\sim$ is

$$\mathcal{C}_{r-1} = \{P_1, P_2, \ldots, P_{r-1}\}$$

such that $P_1 = \{\alpha \in Z_{r-1} : \mathrm{im}\,\alpha = Y \setminus \{1\}\} \cup \{\alpha \in Z_{r-1} : \mathrm{im}\,\alpha = Y \setminus \{r\}\}$ and $P_i = \{\alpha \in Z_{r-1} : \mathrm{im}\,\alpha = Y \setminus \{i\}\}$ for all $2 \le i \le r-1$. Notice that $Z_{r-1} = \bigcup_{i=1}^{r-1} P_i$.

Lemma 4.8 *Let A be any generating set of $ODPI_{n,r}$ and P_i an equivalence class of Z_{r-1} $(1 \le i \le r-1)$. Then $A \cap P_i \neq \emptyset$.*

Proof Let $\alpha = \mathrm{id}_{Y\setminus\{r\}} \in P_1$ or $\alpha = \mathrm{id}_{Y\setminus\{i\}} \in P_i$ for some $2 \le i \le r-1$. Then $\alpha \in Z_{r-1}$ and $1 \in \mathrm{dom}\,\alpha$. Since A is a generating set of $ODPI_{n,r}$, we can write $\alpha = \alpha_1\alpha_2\cdots\alpha_p$ for some $\alpha_1, \alpha_2, \ldots, \alpha_p \in A$. If $|\mathrm{im}\,\alpha_j| \le r-2$ for some $1 \le j \le p$, then $|\mathrm{im}\,\alpha| = |\mathrm{im}\,(\alpha_1\alpha_2\cdots\alpha_p)| \le |\mathrm{im}\,\alpha_j| \le r-2$, a contradiction. Hence, $\alpha_1, \alpha_2, \ldots, \alpha_p \in Z_r \cup Z_{r-1}$. If $\alpha_1, \alpha_2, \ldots, \alpha_p \in Z_r$, then we get $\alpha_1 = \alpha_2 = \cdots = \alpha_p = \mathrm{id}_Y$ by Lemma 4.7. It follows that $\alpha = \mathrm{id}_Y \in Z_r$, this is a contradiction. Thus, there is $\alpha_q \in Z_{r-1}$ for some $1 \le q \le p$. We prove $\mathrm{dom}\,\alpha \subseteq \mathrm{dom}\,\alpha_q$. If $\alpha_q = \alpha_1$, then $\mathrm{dom}\,\alpha = \mathrm{dom}\,(\alpha_q\alpha_2\cdots\alpha_p) \subseteq \mathrm{dom}\,\alpha_q$. But, if $\alpha_q \neq \alpha_1$, we may assume that q is the least integer among $2, 3, \ldots, p$ in which $\alpha_q \in Z_{r-1}$, this means $\alpha_1, \alpha_2, \ldots, \alpha_{q-1} \in Z_r$. We now have $\alpha = \alpha_1\alpha_2\cdots\alpha_{q-1}\alpha_q\cdots\alpha_p$ such that $\alpha_1, \alpha_2, \ldots, \alpha_{q-1} \in Z_r$. Then by Lemma 4.7, we get $\alpha_1 = \alpha_2 = \cdots = \alpha_{q-1} = \mathrm{id}_Y$. It follows that $\alpha = (\mathrm{id}_Y)\alpha_q\cdots\alpha_p$. Suppose that $\mathrm{dom}\,\alpha_q \nsubseteq Y$. Then $|\mathrm{im}\,\alpha| \le |\mathrm{im}(\mathrm{id}_Y\alpha_q)| \le r-2$, which is a contradiction. Whence, $\mathrm{dom}\,\alpha_q \subseteq Y$ and so $\alpha = \alpha_q\alpha_{q+1}\cdots\alpha_p$. This implies that $\mathrm{dom}\,\alpha \subseteq \mathrm{dom}\,\alpha_q$. Since $|\mathrm{dom}\,\alpha| = |\mathrm{dom}\,\alpha_q| = r-1$ is finite, we obtain $\mathrm{dom}\,\alpha = \mathrm{dom}\,\alpha_q$. Thus, $g(\mathrm{im}\,\alpha_q) = g(\mathrm{dom}\,\alpha_q) = g(\mathrm{dom}\,\alpha) = g(\mathrm{im}\,\alpha)$, that is, $\alpha_q \in P_i$. Therefore, $A \cap P_i \neq \emptyset$. □

Lemma 4.9 *For $2 \le r \le n-1$, we have $\mathrm{rank}(ODPI_{n,r}) \ge n$.*

Proof It follows directly from Lemmas 4.6 and 4.8 that $\mathrm{rank}(ODPI_{n,r}) \ge |Z_r| + |\mathcal{C}_{r-1}| = (n-r+1) + (r-1) = n$. □

We define

$$\mu = \begin{pmatrix} 1 & 2 & \cdots & r-1 \\ 2 & 3 & \cdots & r \end{pmatrix} \text{ and } T = \{\mu\} \cup \{\mathrm{id}_{Y\setminus\{i\}} : 2 \le i \le r-1\}.$$

It is known that $ODP_r = \langle T, \mu^{-1} \rangle$ by Lemma 1.3.

Lemma 4.10 *$ODP_r \subseteq \langle Z_r \cup T \rangle$.*

Proof Let δ_1 be defined as above Lemma 4.6. That is,

$$\delta_1 = \begin{pmatrix} 2 & 3 & \cdots & r & r+1 \\ 1 & 2 & \cdots & r-1 & r \end{pmatrix} \in Z_r.$$

We can write $\mu^{-1} = \delta_1\mu\delta_1$, where $\mu \in T$, i.e., $\mu^{-1} \in \langle Z_r \cup T \rangle$. Thus, $T \cup \{\mu^{-1}\} \subseteq \langle Z_r \cup T \rangle$ and so $ODP_r = \langle T, \mu^{-1} \rangle \subseteq \langle Z_r \cup T \rangle$. □

Theorem 4.11 *For $2 \le r \le n-1$, $ODPI_{n,r} = \langle Z_r \cup T \rangle$ and $\mathrm{rank}(ODPI_{n,r}) = n$.*

Proof By Lemma 4.10, we have $ODP_r \subseteq \langle Z_r \cup T\rangle$. We prove $ODPI_{n,r} \setminus ODP_r \subseteq \langle Z_r \cup T\rangle$. Let γ be any element in $ODPI_{n,r} \setminus ODP_r$. Then $\gamma \in Z_k$ for some $1 \le k \le r$. In fact, $\gamma \in P_i \setminus ODP_r$ for some $1 \le i \le |\mathcal{C}_k|$. By the construction of P_i in Z_k, we can write

$$\gamma = \begin{pmatrix} (r-\omega_i)+p & (r-\omega_i+d_1)+p & \cdots & (r-d_{k-1})+p & r+p \\ 1+q & 1+d_1+q & \cdots & 1+(\omega_i-d_{k-1})+q & 1+\omega_i+q \end{pmatrix}$$

for some $1 \le p \le n-r$ and $0 \le q \le r-(1+\omega_i)$, where $(d_1, d_2, \ldots, d_{k-1})$ is the gap of all elements of P_i in Z_k and $\omega_i = \sum_{j=1}^{k-1} d_j$. Observe that $r - \omega_i = r - \sum_{j=1}^{k-1} d_j \ge 1$. We define $\lambda \in P_i \cap ODP_r$ by

$$\lambda = \begin{pmatrix} r-\omega_i & r-(\omega_i-d_1) & \cdots & r-d_{k-1} & r \\ 1+q & 1+d_1+q & \cdots & 1+(\omega_i-d_{k-1})+q & 1+\omega_i+q \end{pmatrix}.$$

It is clear that $r-\omega_i, r-(\omega_i-d_1), \ldots, r-(d_{k-2}+d_{k-1}), r-d_{k-1} \in Y$ such that $1 \le r-\omega_i < r-(\omega_i-d_1) < \cdots < r-(d_{k-2}+d_{k-1}) < r-d_{k-1} < r$. So, we can write $\delta_p \in Z_r$ as follows:

$$\delta_p = \begin{pmatrix} 1+p & \cdots & (r-\omega_i)+p & \cdots & (r-\omega_i+d_1)+p & \cdots & (r-d_{k-1})+p & \cdots & r+p \\ 1 & \cdots & r-\omega_i & \cdots & r-(\omega_i-d_1) & \cdots & r-d_{k-1} & \cdots & r \end{pmatrix}.$$

Then $\gamma = \delta_p\lambda \in \langle Z_r \cup ODP_r\rangle \subseteq \langle Z_r \cup T\rangle$. It follows that $ODPI_{n,r} \setminus ODP_r \subseteq \langle Z_r \cup T\rangle$. Hence, $ODPI_{n,r} = \langle Z_r \cup T\rangle$ and so $\operatorname{rank}(ODPI_{n,r}) \le |Z_r \cup T| = (n-r+1)+(r-1) = n$. By Lemma 4.9, we have $\operatorname{rank}(ODPI_{n,r}) = n$. □

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